COMPOSITION OPERATORS BETWEEN HARDY AND BLOCH-TYPE SPACES OF THE UPPER HALF-PLANE

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Abstract. In this paper, we study composition operators $C_\varphi f = f \circ \varphi$, induced by a fixed analytic self-map of the of the upper half-plane, acting between Hardy and Bloch-type spaces of the upper half-plane.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\varphi$ be a holomorphic self-map of $\mathbb{D}$. Then the equation $C_\varphi f = f \circ \varphi$, for $f$ analytic in $\mathbb{D}$ defines a composition operator $C_\varphi$ with inducing map $\varphi$. During the past few decades, composition operators have been studied extensively on spaces of functions analytic on the open unit disk $\mathbb{D}$. As a consequence of the Littlewood Subordination principle [3] it is known that every analytic self-map $\varphi$ of the open unit disk $\mathbb{D}$ induces a bounded composition operator on Hardy and weighted Bergman spaces of the open unit disk $\mathbb{D}$. However, if we move to Hardy and weighted Bergman spaces of the upper half-plane

$$\pi^+ = \{z \in \mathbb{C} : \text{Im } z > 0\},$$

the situation is entirely different. There do exist analytic self-maps of the upper half-plane, which do not induce composition operators on the Hardy spaces and weighted Bergman spaces of the upper half-plane (see [4], [10] and [12]. Interesting work on composition operators on Hardy spaces of the upper half-plane have been done by Singh [11], Singh and Sharma [12], [13], Sharma [9] and Matache [4] and [5]. Recently, several authors have studied composition operators and weighted composition operators on Bloch-type spaces of functions analytic in the open unit disk $\mathbb{D}$. For example, one can refer to [6] and [7] and the references therein for the study of these operators on Bloch-type spaces. However, composition operators on the Bloch-type spaces of the upper half-plane remain untouched so far. The main theme of this paper is to study composition operators between Hardy and Bloch type spaces of the upper half-plane. The plan of the rest of the paper is as follows. In the next section
we introduce Hardy and Bloch-type spaces of the upper half-plane. Section 3 is devoted to characterize boundedness of composition operators on the Bloch space of the upper half-plane whereas boundedness of composition operators on Growth spaces is tackled in section 4. Sections 5 and 6 deals with the boundedness of composition operators between Hardy and Bloch-type spaces of the upper half-plane.

2. Preliminaries

In this section we review the basic concepts and collect some essential facts that will be needed throughout the paper.

2.1. Hardy spaces of the upper half-plane.

For $1 \leq p < \infty$, the Hardy space of the upper half-plane is defined as

$$H^p(\pi^+) = \{ f : \pi^+ \to \mathbb{C} | f \text{ is analytic and } ||f||_p^p = \sup_{y>0} \int_{-\infty}^\infty |f(x+iy)|^p dx < \infty \}.$$ 

With this norm $H^p(\pi^+)$ becomes a Banach space and for $p = 2$, it is a Hilbert space. To know more about these spaces, we refer to [1] and [2].

The growth of functions in the Hardy space is essential in our study. To this end the following estimate will be useful. For $f \in H^p(\pi^+)$, we have

$$|f(x + iy)|^p \leq ||f||_p^p \frac{1}{2\pi y}. \quad (2.1)$$

2.2. Bloch space of the upper half-plane.

The Bloch space of the upper half-plane $\pi^+$, denoted by $B_\infty(\pi^+)$, is defined to be the space of analytic functions $f$ on $\pi^+$ such that

$$||f||_{B_\infty} = \sup_{z \in \pi^+} \{ \text{Im } |f'(z)| \} < \infty.$$ 

It is easy to check that $||f||_{B_\infty}$ is a complete semi-norm on $B_\infty(\pi^+)$. 

2.3. Growth space of the upper half-plane.

The Growth space of the upper half-plane $\pi^+$, denoted by $A_\infty(\pi^+)$, is defined to be the space of analytic functions $f$ on $\pi^+$ such that

$$||f||_{A_\infty} = \sup_{z \in \pi^+} \{ |f(z)| \} < \infty.$$ 

It is easy to check that $A_\infty(\pi^+)$ is a (non separable) Banach space with the norm defined above.
3. Composition operators on $B_\infty(\pi^+)$

In [4], Matache proved that a linear fractional map

\begin{equation}
\varphi(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0,
\end{equation}

induces a bounded composition operator on Hardy spaces $H^p(\pi^+)$ of the upper half plane if and only if $c = 0$. However, by a simple application of the Schwarz-Pick Theorem in the upper half-plane, we can show that every holomorphic map $\varphi$ of $\pi^+$ such that $\varphi(\pi^+) \subset \pi^+$ induces a bounded composition operator on the Bloch space $B_\infty(\pi^+)$. Let us first state the Schwarz-Pick Theorem in the upper half-plane.

**Schwarz-Pick Theorem in the upper half-plane.** Let $\varphi$ be a holomorphic map of $\pi^+$ such that $\varphi(\pi^+) \subset \pi^+$. Then for all $z_1, z_2 \in \pi^+$,

\[ \left| \frac{\varphi(z_1) - \varphi(z_2)}{\varphi(z_1) - \varphi(z_2)} \right| \leq \frac{|z_1 - z_2|}{|z_1 - z_2|}. \]

Also for all $z \in \pi^+$,

\[ \left| \frac{\varphi'(z)}{\varphi(z)} \right| \leq \frac{1}{\text{Im } z}. \]

Moreover, if equality holds in one of the two inequalities above, then $\varphi$ must be a Mobius transformation with real coefficients. That is, if equality holds, then $\varphi$ is given by (3.1).

**Theorem 3.1.** For any holomorphic map $\varphi$ of $\pi^+$ such that $\varphi(\pi^+) \subset \pi^+$, the composition operator $C_\varphi : B_\infty(\pi^+) \to B_\infty(\pi^+)$ is bounded.

**Proof.** For arbitrary $z \in \pi^+$ and $f \in B_\infty(\pi^+)$

\[ \text{Im } z |(C_\varphi f)'(z)| = \text{Im } z |f'((\varphi(z)))||\varphi'(z)|| \leq \frac{\text{Im } z}{\text{Im } \varphi(z)} |f'||\varphi'(z)|, \]

and, consequently, by a simple application of the Schwarz-Pick Theorem on the upper half-plane,

\[ \sup_{z \in \pi^+} \frac{\text{Im } z}{\text{Im } \varphi(z)} |\varphi'(z)| < 1, \]

we have $C_\varphi f \in B_\infty(\pi^+)$. Hence by an analogue of the Closed Graph Theorem $C_\varphi$ is bounded. \(\square\)

4. Composition operators on $A_\infty(\pi^+)$

**Theorem 4.1.** Let $\varphi$ be a holomorphic map of $\pi^+$ such that $\varphi(\pi^+) \subset \pi^+$. Then $C_\varphi : A_\infty(\pi^+) \to A_\infty(\pi^+)$ is bounded if and only if

\begin{equation}
(4.1) \quad \sup_{z \in \pi^+} \frac{\text{Im } z}{\text{Im } \varphi(z)} < \infty.
\end{equation}
First suppose that (4.1) holds. Then boundedness of $C_\varphi$ on $A_\infty(\pi^+)$ can be proved on similar lines as in the proof of Theorem 3.1.

Conversely, suppose $C_\varphi$ is bounded. Fix a point $z_0 \in \pi^+$ and let $w = \varphi(z_0)$. Consider the function $f_w(z) = 1/(z - w)$. Then $f \in A_\infty(\pi^+)$ and $\|f_w\|_{A_\infty} \leq 1$. Boundedness of $C_\varphi : A_\infty(\pi^+) \rightarrow A_\infty(\pi^+)$ implies that there is a positive constant $C$ such that, for each $z \in \pi^+$ we have $\Im z|f_w(\varphi(z))| \leq C$. In particular take $z = z_0$, we get

$$\frac{\Im z_0}{\Im \varphi(z_0)} \leq 2C.$$ 

Since $z_0 \in \pi$ is arbitrary, the result follows. \hfill \square

**Note.** If $c = a + ib \in \pi^+$ and $\varphi(\pi) = c$ for all $z \in \pi^+$, then $\varphi$ does not induce a bounded composition operator on $A_\infty(\pi^+)$. 

**Corollary 4.2.** Let $\varphi(z) = \frac{az + b}{cz + d}$, $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. Then necessary and sufficient condition that $C_\varphi$ is bounded on $A_\infty(\pi^+)$ is that $c = 0$.

**Proof.** First suppose that $C_\varphi$ is bounded. Then for $z = x + iy \in \pi^+$,

$$\sup_{z \in \pi^+} \frac{\Im z}{\Im \varphi(z)} = \sup_{z \in \pi^+} \frac{(cx + d)^2 + c^2y^2}{(ad - bc)},$$

which is finite only if $c = 0$. Conversely, if $c = 0$, then $\varphi(z) = (a/d)z + (b/d)$, where $ad > 0$ and so

$$\sup_{z \in \pi^+} \frac{\Im z}{\Im \varphi(z)} = \frac{d}{a} < \infty.$$ 

Thus $C_\varphi$ is bounded on $A_\infty(\pi^+)$. \hfill \square

5. Composition operators from $H^p(\pi^+)$ into $A_\infty(\pi^+)$

**Theorem 5.1.** Let $1 \leq p < \infty$ and $\varphi$ be a holomorphic map of $\pi^+$ such that $\varphi(\pi^+) \subset \pi^+$. Then $C_\varphi : H^p(\pi^+) \rightarrow A_\infty(\pi^+)$ is bounded if and only if

$$\sup_{z \in \pi^+} \frac{\Im z}{(\Im \varphi(z))^{1/p}} < \infty.$$ 

**Proof.** First suppose that

$$M = \sup_{z \in \pi^+} \frac{\Im z}{(\Im \varphi(z))^{1/p}} < \infty.$$ 

By (2.1), $|f(z)|^p \leq ||f||^p_{H^p(\pi^+)}$, for all $z = x + iy \in \pi^+$ and $f \in H^p(\pi^+)$. Thus, for $f \in H^p(\pi^+)$

$$||C_\varphi f||_{A_\infty} = \sup_{z \in \pi^+} \Im z|C_\varphi f(z)| \leq \sup_{z \in \pi^+} \frac{\Im z}{(2\pi \Im \varphi(z))^{1/p}} ||f||_p$$

$$= (M/(2\pi))^{1/p} ||f||_p.$$
Hence $C_\varphi : H^p(\pi^+) \to A_\infty(\pi^+)$ is bounded. Conversely, suppose that $C_\varphi : H^p(\pi^+) \to A_\infty(\pi^+)$ is bounded. Fix a point $z_0 \in \pi^+$ and let $w = \varphi(z_0)$. Consider the function
\[ f_w(z) = \frac{(\Im w)^{2-1/p}}{\pi^{1/p}(z - \overline{w})^2}. \]

Then
\[ \|f_w\|_p^p = \sup_{y>0} \int_{-\infty}^{\infty} |f_w(x + iy)|^p \, dx \]
\[ = \frac{(\Im w)^{2p-1}}{\pi} \sup_{y>0} \int_{-\infty}^{\infty} \frac{1}{|z - \overline{w}|^{2p}} \, dx. \]

Writing $w = u + iv$ and $z = x + iy$, we get
\[ |z - \overline{w}|^{2p} \geq (v + y)^{2p-2} ((x - u)^2 + (y + v)^2) \]
and so
\[ \|f_w\|_p^p \leq \frac{v^{2p-1}}{\pi} \sup_{y>0} \frac{1}{(y+v)^{2p-1}} \int_{-\infty}^{\infty} \frac{y+v}{(x-u)^2 + (y+v)^2} \, dx \]
\[ = \frac{v^{2p-1}}{\pi} \sup_{y>0} \frac{1}{(y+v)^{2p-1}} \pi \]
\[ = 1. \]

Boundedness of $C_\varphi : H^p(\pi^+) \to A_\infty(\pi^+)$ implies that there is a positive constant $C$ such that, for each $z \in \pi^+$ we have $\Im z |f_w(\varphi(z))| \leq C$. In particular take $z = z_0$, we get
\[ \frac{\Im z_0}{(\Im \varphi(z_0))^{1/p}} \leq 4\pi^{1/p} C. \]

Since $z_0 \in \pi^+$ is arbitrary, the result follows. □

**Corollary 5.2.** Let $\varphi(z) = \frac{az + b}{cz + d}$, $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. Then $C_\varphi : H^p(\pi^+) \to B_\infty(\pi^+)$ is bounded if and only if $c = 0$ and $p = 1$.

**Proof.** First suppose that $C_\varphi$ is bounded. Then for $z = x + iy \in \pi^+$,
\[ \sup_{z \in \pi^+} \left( \frac{\Im z}{(\Im \varphi(z))^{1/p}} \right) = \sup_{z \in \pi^+} \left( \frac{(cx + d)^2 + c^2 y^2)^{1/p} y}{(ad - bc)^{1/p} y^{1/p}} \right), \]
which is finite only if $c = 0$ and $p = 1$. Conversely, if $c = 0$ and $p = 1$, then
\[ \sup_{z \in \pi^+} \frac{\Im z}{\Im \varphi(z)} = \frac{d}{a} < \infty. \]

Hence $C_\varphi : H^p(\pi^+) \to B_\infty(\pi^+)$ is bounded. □

We next characterize boundedness of composition operators from $H^p(\pi^+)$ into $B_\infty(\pi^+)$. 
6. Composition operators from $H^p(\pi^+)$ into $B_\infty(\pi^+)$

**Theorem 6.1.** Let $1 \leq p < \infty$ and $\varphi$ be a holomorphic map of $\pi^+$ such that $\varphi(\pi^+) \subset \pi^+$. Then $C_\varphi : H^p(\pi^+) \to B_\infty(\pi^+)$ is bounded if and only if

$$\sup_{z \in \pi^+} \frac{\operatorname{Im} z}{(\operatorname{Im} \varphi(z))^{(p+1)/p}} |\varphi'(z)| < \infty. \quad (6.1)$$

**Proof.** First suppose that (6.1) holds. Let $f \in H^p(\pi^+)$. Then by Cauchy integral formula in $\pi^+$ [1], we have

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{(t - z)} \, dt, \quad z = x + iy \in \pi^+.$$

Thus

$$|f'(z)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|f(t)|}{|t - z|^2} \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|f(t)|}{(t - x)^2 + y^2} \, dt.$$

Since

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{((t - x)^2 + y^2)} \, dt = 1,$$

$x^p$ is convex, we have by Jensen’s inequality [8], p 62,

$$|f'(z)|^p \leq \frac{1}{2\pi y^{p-1}} \int_{-\infty}^{\infty} \frac{|f(t)|^p}{((t - x)^2 + y^2)} \, dt \leq \frac{1}{2\pi y^{p+1}} \int_{-\infty}^{\infty} |f(t)|^p \, dt.$$

Thus

$$|f'(z)|^p \leq \frac{||f||_p^p}{2\pi y^{p+1}}.$$

Thus, for $f \in H^p(\pi^+)$

$$||C_\varphi f||_{B_\infty} = \sup_{z \in \pi^+} \operatorname{Im} z |(C_\varphi f)'(z)| \leq \sup_{z \in \pi^+} \operatorname{Im} z / (2^p \operatorname{Im} \varphi(z))^{(p+1)/p} |\varphi'(z)| \, ||f||_p$$

$$= M ||f||_p.$$

Hence $C_\varphi : H^p(\pi^+) \to B_\infty(\pi^+)$ is bounded. Conversely, suppose that $C_\varphi : H^p(\pi^+) \to B_\infty(\pi^+)$ is bounded. Fix a point $z_0 \in \pi^+$ and let $w = \varphi(z_0)$. Consider the function

$$f_w(z) = \frac{(\operatorname{Im} w)^{2-1/p}}{\pi^{1/p} (z - w)^2}.$$
Then
\[ f'_w(z) = \frac{(\text{Im } w)^{2-1/p}}{\pi^{1/p}(z - \overline{w})^3}. \]

As in Theorem 5.1, we have \( ||f_w||_p \leq 1 \). Boundedness of \( C_\varphi : H^p(\pi^+) \to \mathcal{B}_\infty(\pi^+) \) implies that there is a positive constant \( C \) such that \( ||C_\varphi f||_{\mathcal{B}_\infty} \leq C ||f_w||_p \leq C \). Hence, for each \( z \in \pi^+ \)
\[ \text{Im } z |f'_w(\varphi(z))\varphi'(z)| \leq C. \]

In particular, putting \( z = z_0 \), we get
\[ \frac{\text{Im } z_0 |\varphi'(z_0)|}{(\text{Im } \varphi(z_0))^{(p+1)/p}} < 4\pi^{1/p}C. \]

Since \( z_0 \in \pi^+ \) is arbitrary, the result follows. \( \square \)

**Corollary 6.2.** Let \( \varphi(z) \) be a holomorphic self-map of \( \pi^+ \) given by (3.2). Then \( C_\varphi : H^p(\pi^+) \to \mathcal{B}_\infty(\pi^+) \) is not bounded.

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